Math 112: Introductory Real Analysis

Last time:
- $f: X \to Y$ continuous, $K \subseteq X$ compact $\Rightarrow f(K) \subseteq Y$ compact
- uniform continuity

Today: Continuity and connectedness, discontinuity

Recall: $E \subseteq X$ is connected if there are no nonempty subsets $A, B \subset E$ such that $A \cup B = E$,
$\bar{A} \cap B = \phi$,
$A \cap \bar{B} = \phi$

Thm If $f: X \to Y$ is a continuous map between metric spaces and if $E \subseteq X$ is connected, then $f(E) \subseteq Y$ is connected.

proof) Assume, on the contrary, that $f(E) = A \cup B$ where $A, B \subset Y$ are nonempty subsets such that $\bar{A} \cap B = \phi$, $A \cap \bar{B} = \phi$.

Put $C := E \cap f^{-1}(A)$ and $D := E \cap f^{-1}(B)$.

Then $E = C \cup D$, and both C and D are nonempty.
(since $A = f(C)$ and $B = f(D)$)

Moreover, since $f(C) = A \subseteq \bar{A}$, we have $C \subseteq f^{-1}(\bar{A})$, where the latter set is closed (since f is continuous).

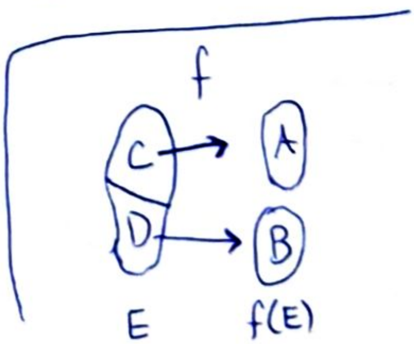
Hence $\bar{C} \subseteq f^{-1}(\bar{A})$, and $f(\bar{C}) \subseteq \bar{A}$.

Since $f(D) = B$ and $\bar{A} \cap B = \phi$, we conclude that $\bar{C} \cap D = \phi$.

The same argument shows that $C \cap \bar{D} = \phi$.

This is impossible if E is connected.

Therefore, $f(E)$ must be connected. ■



Cor (Intermediate value theorem)

Let f be a continuous real function on the interval $[a,b]$.

If $f(a) < f(b)$ and c is a number such that $f(a) < c < f(b)$,

then there exists a point $x \in (a,b)$ such that $f(x) = c$.

proof) Since $[a,b]$ is connected, $f([a,b]) \subset \mathbb{R}$ is connected,

and hence $c \in [f(a), f(b)] \subseteq f([a,b])$. The assertion follows. ■

- Discontinuities

Def Let f be a function defined on (a,b). Consider any point $x \in [a,b)$.

We write $f(x+) = q$ if $f(t_n) \to q$ as $n \to \infty$, for all sequences $\{t_n\}$ in (x,b) such that $t_n \to x$.

In that case, we say q is the right-hand limit of f at x.

Likewise, for any $x \in (a,b]$, the left-hand limit $f(x-)$ is defined using sequences $\{t_n\}$ in (a,x).

Rmk It is clear that for any point $x \in (a,b)$, $\lim_{t \to x} f(t)$ exists if and only if

$$f(x+) = f(x-) \quad \left(= \lim_{t \to x} f(t)\right).$$

Def Let f be a function on (a,b) that is discontinuous at $x \in (a,b)$.

If $f(x+)$ and $f(x-)$ exist, we say f has a discontinuity of the first kind (or a simple discontinuity) at x.

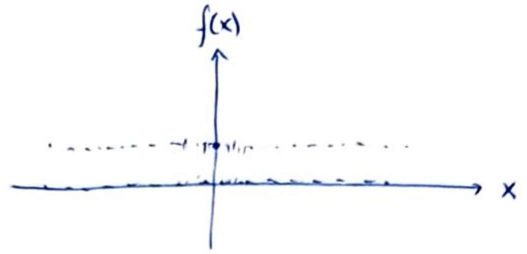
Otherwise, we say f has a discontinuity of the second kind at x.

3/

Examples

(a)

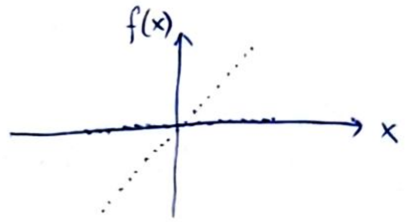
$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R}\setminus\mathbb{Q} \end{cases}$$



Then f has a discontinuity of the second kind at every point $x \in \mathbb{R}$.

(b)

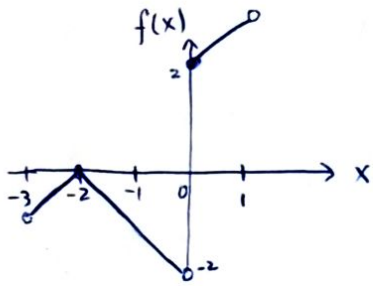
$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R}\setminus\mathbb{Q} \end{cases}$$



Then f is continuous at $x=0$
and has discontinuity of the second kind at every other point.

(c)

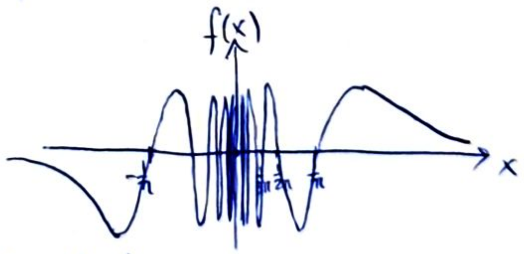
$$f(x) = \begin{cases} x+2 & \text{if } x \in (-3,-2) \\ -x-2 & \text{if } x \in [-2,0) \\ x+2 & \text{if } x \in [0,1) \end{cases}$$



Then f has a simple discontinuity at $x=0$
and is continuous at every other point of $(-3,1)$.

(d)

$$f(x) = \begin{cases} \sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$



Then f has a discontinuity of the second kind at $x=0$
and is continuous at every other point.

<u>Def</u> A real function f on (a,b) is said to be <u>(monotonically) increasing</u> if $a<x<y<b$ implies $f(x) \leq f(y)$.

If the last inequality is reversed, we say f is <u>(monotonically) decreasing</u>.

<u>Thm</u> Let f be a monotonically increasing function on (a,b).

Then $f(x+)$ and $f(x-)$ exist at every point $x \in (a,b)$, and

$$\sup_{a<t<x} f(t) = f(x-) \leq f(x) \leq f(x+) = \inf_{x<t<b} f(t).$$

proof) By hypothesis, $\{f(t) \mid a<t<x\}$ is bounded above by $f(x)$, and therefore $\sup_{a<t<x} f(t)$ exists. Clearly, $\sup_{a<t<x} f(t) \leq f(x)$. We need to show $\sup_{a<t<x} f(t) = f(x-)$.

Let $\varepsilon>0$ be given. Then there exists $\delta>0$ such that

$$a < x-\delta < x \quad \text{and} \quad \sup_{a<t<x} f(t) - \varepsilon < f(x-\delta) \leq \sup_{a<t<x} f(t).$$

(Otherwise $\sup_{a<t<x} f(t) - \varepsilon$ is an upper bound of $\{f(t) \mid a<t<x\}$)

Since f is monotonic,

$$f(x-\delta) \leq f(t) \leq \sup_{a<t'<x} f(t') \quad \text{for all } t \in (x-\delta, x).$$

Hence,

$$\left| f(t) - \sup_{a<t'<x} f(t') \right| < \varepsilon \quad \text{for all } t \in (x-\delta, x),$$

and therefore $f(x-) = \sup_{a<t<x} f(t)$.

That $f(x+) = \inf_{x<t<b} f(t)$ can be proved in a similar way. ■

<u>Cor</u> Monotonic functions have no discontinuities of the second kind.

<u>Cor</u> If f is monotonic on (a,b), then the set of points of (a,b) at which f is discontinuous is at most countable.

proof) At every discontinuity x, associate a rational number $r(x)$ s.t. $f(x-) < r(x) < f(x+)$. ■